

HOPFISH STRUCTURE AND MODULES OVER IRRATIONAL ROTATION ALGEBRAS

CHRISTIAN BLOHMANN, XIANG TANG, AND ALAN WEINSTEIN

ABSTRACT. Inspired by the group structure on S^1/\mathbb{Z} , we introduce a weak hopfish structure on an irrational rotation algebra A of finite Fourier series. We consider a class of simple A -modules defined by invertible elements, and we compute the tensor product between these modules defined by the hopfish structure. This class of simple modules turns out to generate an interesting commutative unital ring.

1. INTRODUCTION

The starting point for this work is the following two principles.

- (1) If H is a group, many algebras of functions on H (under pointwise multiplication) are Hopf algebras with the coproduct $\Delta(a)(g, h) = a(gh)$.
- (2) If a group K acts on a space X , then an algebra which is a crossed product of K acting on an algebra of functions on X is a good substitute for an algebra of functions on X/K when X/K is badly behaved as a topological space.

Now let $X = S^1$ be the unit circle of complex numbers, or $U(1)$, and let $K = \mathbb{Z}$ be the subgroup generated by an element $e^{i\lambda}$, where λ is an irrational multiple of 2π . Since the quotient space $H = S^1/\mathbb{Z}$ is a group, the two principles above suggest that the crossed product built from \mathbb{Z} acting on an algebra of functions on S^1 via rotation through the angle λ , which we will call irrational rotation algebra, should be something like a Hopf algebra. But it is well known that an irrational rotation algebra is not a Hopf algebra; it does not even admit a counit, i.e. a unital homomorphism to \mathbb{C} . (All of our function spaces will be complex valued.) As a remedy, the notion of **hopfish algebra** was introduced in [9], based on the subtext of item 2 above to the effect that the substitution of a crossed product algebra for an algebra of functions on a quotient is based on the theory of Morita equivalence, in which bimodules are interpreted as generalized homomorphisms between algebras. Thus, the coproduct and counit of a hopfish algebra A are taken to be $(A \otimes A, A)$ - and (\mathbb{C}, A) -bimodules rather than algebra homomorphisms. The antipode is still an ordinary antihomomorphism from A to itself; we defer the precise definition to Section 3.

In [9] we stopped short of dealing with irrational rotation algebras, since it appeared that one would need to work in the world of C^* -algebras, which was beyond

Date: Version: April 12, 2006.

2000 *Mathematics Subject Classification.* 16W30(Primary), 16D60(Secondary).

Key words and phrases. hopfish algebra, groupoid, bimodule, quantum torus, cyclic module.

Christian Blohmann was supported by a Marie Curie Fellowship of the European Union; Alan Weinstein was partially supported by NSF grant DMS-0204100.

the purely algebraic scope of that paper. In fact, an algebraic treatment turns out to be possible and quite interesting, even if it may not be optimal. In the present article, we discuss the hopfish structure of a crossed product $*$ -algebra built from the irrational rotation action of the integers on the algebra of “polynomial” functions on the circle (i.e. on the algebra of finite Fourier series under pointwise multiplication). Actually, the structure is not quite hopfish—we need a slight weakening of the antipode criterion, but otherwise, everything works as in [9].

The notion of hopfish algebra is a dualization of that of “group object in the category of differentiable stacks”. In the geometric language of [2], [3], [7], and [10], a stack is represented by a groupoid (with equivalent groupoids representing the same stack), and a morphism between stacks is represented by a groupoid bibundle. Thus, the group S^1/\mathbb{Z} is considered as a stack represented by the action groupoid $G = \mathbb{Z} \times S^1$. Ignoring that the quotient topology on the set S^1/\mathbb{Z} is trivial, we may construct an equivalence bibundle between S^1/\mathbb{Z} and G . Formal composition with this bibundle and its inverse turns the ordinary group multiplication map from $S^1/\mathbb{Z} \times S^1/\mathbb{Z}$ to S^1/\mathbb{Z} into a perfectly good $(G \times G, G)$ -bibundle B representing the product operation on the stack represented by G . When we dualize, G is replaced by the groupoid algebra A , which is just the irrational rotation algebra. $G \times G$ is then replaced by $A \otimes A$, and B becomes an $(A \otimes A, A)$ -bimodule which is the coproduct of our hopfish algebra structure on A . (All unsubscripted tensor products are taken over \mathbb{C} .)

The construction above is presented in detail in Section 2, along with a construction for the counit of A , derived in a similar way from the inclusion of the identity element into S^1/\mathbb{Z} . In Section 3.2, we discuss the antipode of A , which should be derived from the inversion map on S^1/\mathbb{Z} . Here it turns out that the object which we construct does not quite satisfy the definition in [9] because of difficulties with the duality of infinite-dimensional vector spaces. Therefore, we weaken the notion of antipode to accommodate this example.

In the last part of the paper, we use the coproduct bimodule to construct a tensor product operation on the collection of isomorphism classes of A -modules, and we study the behavior of this application when applied to a nice class of cyclic modules. From the resulting algebraic structure, it is possible to reconstruct the quotient group S^1/\mathbb{Z} .

It is important to note that, although the irrational rotation algebra may be viewed as a deformation of the algebra of functions on a 2-dimensional torus [8], our hopfish structure is *not* a deformation of the Hopf structure associated with the group structure on the torus. Rather, the classical limit of our hopfish structure is a second symplectic groupoid structure on $T^*\mathbb{T}^2$ which is compatible with the one described in [11], whose quantization is the multiplication in the irrational rotation algebra. We thus seem to have a symplectic double groupoid which does not arise from a Poisson Lie group as do those in [6] and [13]. It is possible that such double groupoids are in general the classical limit of hopfish algebras and thus represent a useful generalization of Poisson Lie groups. We would like to also mention that the hopfish algebra structure on an irrational rotation algebra is closely related to the para-Hopf algebroid structure introduced in [5]. In general, there seem to be interesting connections between the notions of hopfish algebra and Hopf algebroid. We hope to pursue these issues in the future.

From the viewpoint of higher algebra, a groupoid with compatible group structure can be viewed as a special case of a 2-groupoid over a point [1]. In this language the groupoid multiplication on $S^1 \times \mathbb{Z}$ is the vertical composition of 2-morphisms, whereas the product group multiplication on $S^1 \times \mathbb{Z}$ is the additional horizontal composition. Accordingly, on the space of functions on $S^1 \times \mathbb{Z}$ both compositions yield convolution products and both inverses yield star structures. The vertical convolution product is the usual convolution of the groupoid algebra which encodes the “bad” topology of the stack it represents. The horizontal convolution is the group convolution which encodes the additional group structure. A preliminary study of this approach for the irrational rotation algebra is very promising: The group convolution operation on the states corresponding to the cyclic modules we consider here is closely related to the tensor product obtained by the hopfish structure. Moreover, the weak hopfish antipode turns out to be the composition of the vertical and the horizontal star involution. This is intriguing and suggests further research.

There are several other important steps in the study of hopfish algebras which we have not yet taken. We still work in the category in which objects are algebras and morphisms are isomorphism classes of bimodules, rather than in the 2-category in which morphisms are bimodules and 2-morphisms are bimodule isomorphisms. As a result, we do not make the modules over A into a tensor category, but simply work with the isomorphism classes of these modules. Second, we are still working purely algebraically, rather than in the world of C^* -algebras, where the use of topological tensor products may allow us to circumvent the weak property of the antipode. Here we should note, though, that even for Hopf algebras, where the coproduct is a homomorphism rather than a bimodule, the extension of the theory to include topologies is highly nontrivial, e.g. [4, 12].

Acknowledgements. We would like to thank Marc Rieffel, Noam Shomron, and Chenchang Zhu for helpful discussions, and Yvette Kosmann-Schwarzbach for comments on the manuscript.

2. CONSTRUCTION OF THE HOPFISH ALGEBRA

We begin by constructing for the action groupoid $S^1 \times \mathbb{Z}$ the bibundles which encode the group multiplication, unit, and inversion of the quotient group S^1/\mathbb{Z} . Spaces of functions on these bibundles are the bimodules giving the (weak) hopfish structure on the irrational rotation algebra.

2.1. The action groupoid and its groupoid algebra. As a set, the action groupoid is¹ $G = S^1 \times \mathbb{Z}$. The left and right projections onto the base S^1 are defined by $\alpha(\theta, k) = \theta + \lambda k$ and $\beta(\theta, k) = \theta$. If we think of a groupoid element as an arrow pointing from an element θ of S^1 to its image under the action of $k \in \mathbb{Z}$, then β is the source map and α the target. When $\beta(\theta_1, k_1) = \alpha(\theta_2, k_2)$, that is, when $\theta_1 = \theta_2 + \lambda k_2$, the product of the two groupoid elements is $(\theta_2, k_1 + k_2)$. The identity bisection is given by the natural embedding $S^1 \hookrightarrow S^1 \times \mathbb{Z}$, $\theta \mapsto (\theta, 0)$. Inversion is the mapping $(\theta, k) \mapsto (\theta - \lambda k, -k)$.

With the product topology and differentiable structure, $G = S^1 \times \mathbb{Z}$ is a smooth, étale groupoid. The family of counting measures on the source fibres is a natural

¹We follow the convention of operator algebra theory, in which the group in a crossed product is written to the right, even though it is acting from the left.

Haar system, leading to an associative convolution product on the space $C_c^\infty(G)$ of smooth, compactly supported functions, defined by

$$(1) \quad (a * b)(\theta, k) := \sum_{k' \in \mathbb{Z}} a(\theta + \lambda k', k - k') b(\theta, k').$$

For the functions

$$a_{nl} \in C(S^1 \times \mathbb{Z}), \quad a_{nl}(\theta, k) := e^{in\theta} \delta_{lk}, \quad n, l \in \mathbb{Z},$$

the convolution product (1) is given by the simple formula

$$(2) \quad a_{n_1 l_1} * a_{n_2 l_2} = e^{i\lambda n_1 l_2} a_{n_1 + n_2, l_1 + l_2}.$$

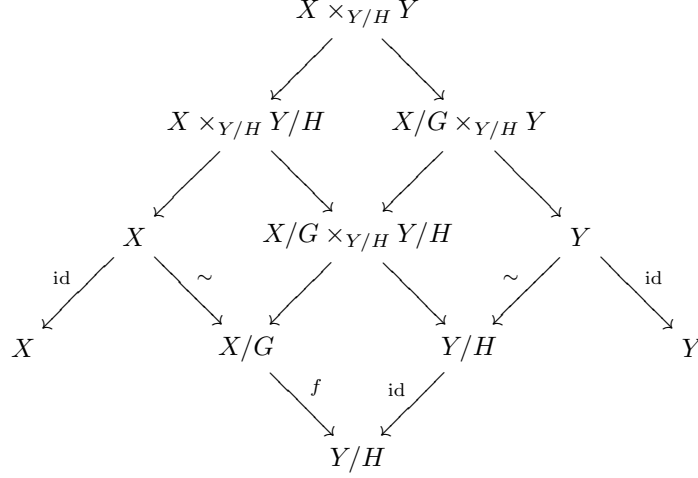
Furthermore, we have a conjugate linear involution defined by $a^*(g) = \overline{a(g^{-1})}$, which yields

$$(3) \quad a_{nl}^* = e^{i\lambda nl} a_{-n, -l}.$$

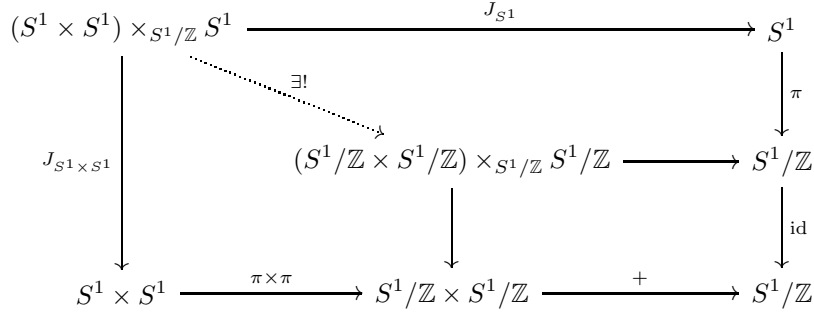
Note, that all a_{nl} are unitary and that a_{00} is the unit element. The vector space spanned by $\{a_{nl} \mid n, l \in \mathbb{Z}\}$ with product (2) and involution (3) is the unital $*$ -algebra generated by a_{10} and a_{01} . It is on this algebra, dense in the irrational rotation C^* -algebra, that we will focus attention in this paper. Denoting it by A , we may think of it as the algebra of “regular functions on an algebraic quantum torus.”

2.2. The coproduct. We wish to construct bibundles which represent for the groupoid $S^1 \times \mathbb{Z}$ the mappings representing the group structure on S^1/\mathbb{Z} . These are obtained by composing the mappings with the bibundle (and its inverse) representing the equivalence between the action groupoid and the quotient space. Even though the quotient space has a “bad topology,” the final bibundles will be perfectly nice.

The general scheme for our constructions is the following. Let G be a groupoid over a set X , H a groupoid over Y , and $f : X/G \rightarrow Y/H$ a morphism between the quotients. Composing the bibundle representing f with the bibundles representing the equivalences between the groupoids and their quotient spaces amounts to filling out the following array of pullback diagrams, working from the given diagrams at the bottom to the final diagram $X \leftarrow X \times_{Y/H} Y \rightarrow Y$.



Now let $\pi : S^1 \rightarrow S^1/\mathbb{Z}$ denote the canonical epimorphism and $+$ the group operation on S^1/\mathbb{Z} . We have the following commutative diagram of two nested pull-back squares, which is a subdiagram of the previous one, tilted by 45 degrees.



The inner pull-back $(S^1/\mathbb{Z} \times S^1/\mathbb{Z}) \times_{S^1/\mathbb{Z}} S^1/\mathbb{Z}$ is simply the graph of addition on S^1/\mathbb{Z} , the pull-back projections being the range and image maps. Any object in the left upper corner which makes the diagram commute can be viewed as a lift of $\text{graph}(+)$ to S^1 . The diagonal arrow is the unique map which exists by the universal property of the inner pull-back. For example, the graph of addition on S^1 is such a lift of $\text{graph}(+)$. The left upper corner of the outer pull-back square can then be viewed as the universal lift into which all other lifts map uniquely.

Let us determine this universal lift explicitly. Denote the image of an element $\theta \in S^1$ under the canonical epimorphism by $[\theta]$. The pull-back is the set

$$(S^1 \times S^1) \times_{S^1/\mathbb{Z}} S^1 = \{(\theta_1, \theta_2, \theta) \in (S^1 \times S^1) \times S^1 \mid [\theta_1] + [\theta_2] = [\theta]\}$$

together with the pull-back projections

$$J_{S^1 \times S^1}(\theta_1, \theta_2, \theta) = (\theta_1, \theta_2), \quad J_{S^1}(\theta_1, \theta_2, \theta) = \theta.$$

It is convenient to identify the pull-back as a set with $S^1 \times S^1 \times \mathbb{Z}$ by the map

$$\begin{aligned} (S^1 \times S^1) \times_{S^1/\mathbb{Z}} S^1 &\xrightarrow{\cong} S^1 \times S^1 \times \mathbb{Z} \\ (\theta_1, \theta_2, \theta) &\mapsto (\theta_1, \theta_2, (\theta - \theta_1 - \theta_2)/\lambda). \end{aligned}$$

For an element $(\theta_1, \theta_2, k) \in S^1 \times S^1 \times \mathbb{Z}$ the pull-back projections are

$$J_{S^1 \times S^1}(\theta_1, \theta_2, k) = (\theta_1, \theta_2), \quad J_{S^1}(\theta_1, \theta_2, k) = \theta.$$

From the left and right groupoid actions of G on S^1 the pull-back inherits a left action of $G \times G$ and a right action of G , the pull-back projections being the momentum maps of these actions. Explicitly, the left action of $(\phi_1, l_1, \phi_2, l_2) \in G \times G$ reads

$$(\phi_1, l_1, \phi_2, l_2) \cdot (\theta_1, \theta_2, k) = (\theta_1 + \lambda l_1, \theta_2 + \lambda l_2, k - l_1 - l_2),$$

which is defined if $\phi_1 = \theta_1$ and $\phi_2 = \theta_2$. The right action of $(\phi, l) \in G$ reads

$$(\theta_1, \theta_2, k) \cdot (\phi, l) = (\theta_1, \theta_2, k - l),$$

which is defined if $\theta_1 + \theta_2 + \lambda k = \phi + \lambda l$. Together with these actions the pull-back becomes a groupoid bibundle.

The space of smooth functions on this groupoid bibundle can now be equipped with the structure of a bimodule over the groupoid algebras: Let $m \in C^\infty(S^1 \times S^1 \times \mathbb{Z})$, let $a \in A \subset C_c^\infty(G)$, and let $x \in S^1 \times S^1 \times \mathbb{Z}$. Then

$$(4) \quad (m \cdot a)(x) := \sum_{g \in \beta^{-1}(J_{S^1}(x))} m(x \cdot g^{-1}) a(g)$$

defines a right action of the groupoid algebra A .

The analogous construction on the left side yields a left $A \otimes A$ -action such that the space of functions on the bibundle becomes an $(A \otimes A, A)$ -bimodule. Again, we are interested only in the algebraic picture so we choose a set of functions

$$(5) \quad d_{n_1 n_2 l} \in C(S^1 \times S^1 \times \mathbb{Z}), \quad d_{n_1 n_2 l}(\theta_1, \theta_2, k) := e^{in_1 \theta_1} e^{in_2 \theta_2} \delta_{lk}, \quad n_1, n_2, l \in \mathbb{Z}.$$

The right action (4) of the basis of A on these function is easily computed to be

$$(6) \quad d_{n_1 n_2 l} \cdot a_{mj} = e^{i\lambda m(l-j)} d_{n_1+m, n_2+m, l-j}.$$

For the left action of $A \otimes A$ we obtain

$$(7) \quad (a_{m_1 j_1} \otimes a_{m_2 j_2}) \cdot d_{n_1 n_2 l} = e^{-i\lambda[(n_1+m_1)j_1 + (n_2+m_2)j_2]} d_{n_1+m_1, n_2+m_2, l-j_1-j_2}.$$

The $(A \otimes A, A)$ -bimodule Δ spanned by $\{d_{n_1 n_2 l}\}$ is a natural candidate for the hopfish coproduct on A .

2.3. The counit. The unit element of the group S^1/\mathbb{Z} can be viewed as a map $e : \{\text{pt}\} \rightarrow S^1/\mathbb{Z}$, $e(\text{pt}) = 0$. The pull-back corresponding to the universal lift to S^1 , is the set

$$\{(\text{pt}, \theta) \in \{\text{pt}\} \times S^1 \mid e(\text{pt}) = [\theta]\} \cong \mathbb{Z}$$

with the right pull-back projection $J_{S^1}(k) = \lambda k$. The right action of $(\phi, l) \in G$ is

$$k \cdot (\phi, l) = k - l,$$

which is defined if $\lambda k = \phi + \lambda l$.

Once more, we choose a set of functions on this right groupoid bundle,

$$(8) \quad e_l \in C(\mathbb{Z}), \quad e_l(k) := \delta_{lk}, \quad l \in \mathbb{Z}.$$

The right action of A is computed to be

$$e_l \cdot a_{mj} = e^{i\lambda m(l-j)} e_{l-j}.$$

The right A -module ϵ spanned by $\{e_l\}$ by is the natural candidate for the hopfish counit of A .

2.4. The antipode. The pull-back corresponding to the universal lift of the inversion map on S^1/\mathbb{Z} to S^1 , is

$$\{(\theta_1, \theta_2) \in S^1 \times S^1 \mid [-\theta_1] = [\theta_2]\} \cong S^1 \times \mathbb{Z},$$

where the identification is $(\theta_1, \theta_2) \mapsto (-\theta_2, (\theta_1 + \theta_2)/\lambda)$. The pull-back projections are $J_{S^1}^{\text{left}}(\theta, k) = \theta + \lambda k$ on the left and $J_{S^1}^{\text{right}}(\theta) = -\theta$ on the right. In light of the axioms of a hopfish algebra, however, we have to view the pull-back as a left $S^1 \times S^1$ bundle with the bundle projection $J_{S^1}^{\text{left}} \times J_{S^1}^{\text{right}}$. The corresponding left action of $(\phi_1, l_1, \phi_2, l_2) \in G \times G$ is

$$(\phi_1, l_1, \phi_2, l_2) \cdot (\theta, k) = (\theta - \lambda l_2, k + l_1 + l_2),$$

which is defined if $\phi_1 = \theta + \lambda k$ and $\phi_2 = -\theta$.

As a set, the pull-back of the graph of the inverse is isomorphic to the groupoid. This suggests choosing as basis for the bimodule the same set of functions as for the groupoid algebra:

$$s_{nl} \in C(S^1 \times \mathbb{Z}), \quad s_{nl}(\theta, k) := e^{in\theta} \delta_{lk}, \quad n, l \in \mathbb{Z}.$$

The left action of $A \otimes A$ on these functions is computed to be

$$(9) \quad (a_{m_1 j_1} \otimes a_{m_2 j_2}) \cdot s_{nl} = e^{i\lambda(m_1 l)} e^{i(n+m_1-m_2)j_2} s_{n+m_1-m_2, l+j_1+j_2}.$$

The left $A \otimes A$ -module \mathcal{S} spanned by $\{s_{nl}\}$ by is the natural candidate for the hopfish antipode.

3. VERIFICATION OF THE AXIOMS

In this section, we study relations among the bimodules $(\Delta, \epsilon, \mathcal{S})$ defined in Section 2.

3.1. The sesquiunital sesquialgebra. When algebra homomorphisms are replaced by bimodules, the notion of unital bialgebra becomes that of sesquiunital sesquialgebra [9].

Definition 1. A sesquiunital sesquialgebra over a commutative ring k is a unital k -algebra A equipped with an $(A \otimes A, A)$ -bimodule Δ (the **coproduct**) and a (k, A) -module (i.e. a right A -module) ϵ (the **counit**), satisfying the following properties.

- (H1) (coassociativity) The $(A \otimes A \otimes A, A)$ -bimodules $(A \otimes \Delta) \otimes_{A \otimes A} \Delta$ and $(\Delta \otimes A) \otimes_{A \otimes A} \Delta$ are isomorphic.
- (H2) (counit) The $(k \otimes A, A) = (A \otimes k, A) = (A, A)$ -bimodules $(\epsilon \otimes A) \otimes_{A \otimes A} \Delta$ and $(A \otimes_k \epsilon) \otimes_{A \otimes A} \Delta$ are both isomorphic to A .

Proposition 1. Let A be the algebra defined in Section 2.1. The coproduct Δ and counit ϵ spanned by the bases in Eq. (5) and Eq. (8) define a sesquiunital sesquilinear algebra structure on A .

Proof. We verify the coassociativity (H1) for Δ ; the proof for the counit property (H2) is similar.

Since A is free of rank one over itself, the bimodule $(A \otimes \Delta) \otimes_{A \otimes A} \Delta$ is the linear span of elements of the form

$$(1 \otimes d_{n_1, n_2, l}) \otimes_{\mathbb{C} \otimes A} d_{n'_1, n'_2, l'}, \quad n_1, n_2, l, n'_1, n'_2, l' \in \mathbb{Z}.$$

By Eqs. (6) and (7),

$$\begin{aligned} & e^{i\lambda m(l-j)}(1 \otimes d_{n_1+m, n_2+m, l-j}) \otimes d_{n'_1, n'_2, l'} \\ &= e^{-i\lambda(n'_2+m)j}(1 \otimes d_{n_1, n_2, l}) \otimes d_{n'_1, n'_2+m, l'-j}. \end{aligned}$$

This relation tells us that the bimodule $(A \otimes \Delta) \otimes_{A \otimes A} \Delta$ is spanned (over \mathbb{C}) by the elements

$$(1 \otimes d_{n_1, n_2, 0}) \otimes d_{n_3, 0, l}, \quad n_1, n_2, n_3, l \in \mathbb{Z}.$$

It is easy to see that these generators are linearly independent and form a basis of $(A \otimes \Delta) \otimes_{A \otimes A} \Delta$.

The left A -module structure on $(A \otimes \Delta) \otimes_{A \otimes A} \Delta$ is computed as follows.

$$\begin{aligned} & (1 \otimes d_{n_1, n_2, 0}) \otimes d_{n_3, 0, l} \cdot a_{mj} \\ &= (1 \otimes d_{n_1, n_2, 0}) \otimes e^{i\lambda m(l-j)} d_{n_3+m, m, l-j} \\ &= e^{i\lambda m(l-j)} (1 \otimes d_{n_1+m, n_2+m, 0}) \otimes d_{n_3+m, 0, l-j}. \end{aligned}$$

And the right $A \otimes A \otimes A$ -module structure on $(A \otimes \Delta) \otimes_{A \otimes A} \Delta$ is computed as follows.

$$\begin{aligned} & (a_{m_1, j_1} \otimes a_{m_2, j_2} \otimes a_{m_3, j_3}) \cdot (1 \otimes d_{n_1, n_2, 0}) \otimes d_{n_3, 0, l} \\ &= e^{-i\lambda\Theta} (1 \otimes d_{n_1+m_2, n_2+m_3, -j_2-j_3}) \otimes d_{n_3+m_1, 0, l-j_1} \\ &= e^{-i\lambda\Theta} (1 \otimes d_{n_1+m_2, n_2+m_3, 0}) \otimes d_{n_3+m_1, 0, l-j_1-j_2-j_3}, \end{aligned}$$

where $\Theta = (n_3 + m_1)j_1 + (n_1 + m_2)j_2 + (n_2 + m_3)j_3$.

A similar computation shows that $(\Delta \otimes A) \otimes_{A \otimes A} \Delta$ has a basis

$$(d_{n_1, n_2, 0} \otimes 1) \otimes d_{0, n_3, l}, \quad n_1, n_2, n_3, l \in \mathbb{Z}.$$

We define the following map $I : (A \otimes \Delta) \otimes_{A \otimes A} \Delta \rightarrow (\Delta \otimes A) \otimes_{A \otimes A} \Delta$ by

$$I((1 \otimes d_{n_1, n_2, 0}) \otimes d_{n_3, 0, l}) = (d_{n_3, n_1, 0} \otimes 1) \otimes d_{0, n_2, l}, \quad n_1, n_2, n_3, l \in \mathbb{Z}.$$

It is easy to check that I is a bimodule isomorphism. \square

3.2. The antipode. We recall from [9] the definition of an antipode for a hopfish algebra.

Definition 2. A preantipode for a sesquiunital sesquialgebra A over k is a left $A \otimes A$ -module \mathcal{S} together with an isomorphism of its k -dual with the right $A \otimes A$ -module $\text{Hom}_A(\epsilon, \Delta)$ of left A -linear maps to ϵ from Δ .²

If a preantipode \mathcal{S} , considered as an (A, A^{op}) -bimodule, is a free left A -module of rank 1, we call \mathcal{S} an **antipode** and say that A along with \mathcal{S} is a **hopfish algebra**.

The definition of (pre)antipode can reformulated as the following two conditions.

- (H3) (preantipode) The dual module \mathcal{S}^* and the space of right A -linear maps $\text{Hom}_A(\epsilon, \Delta)$ are isomorphic as right $A \otimes A$ -modules.
- (H4) (antipode) As a left $A = A \otimes \mathbb{C}$ -module, \mathcal{S} is free of rank one.

When the left $A \otimes A$ -module \mathcal{S} is viewed as an (A, A^{op}) -bimodule, Axiom (H4) states that \mathcal{S} is the modulation of a homomorphism of algebras $S : A^{\text{op}} \rightarrow A$. In fact, this is easily verified. From Eq. (9) we get

$$(a_{mj} \otimes 1) \cdot s_{00} = s_{mj},$$

²We use the convention that homomorphisms map from right to left in order to be consistent with [9].

which shows that s_{00} is a basis of \mathbf{S} as an $A \otimes \mathbb{C}$ -module. Furthermore,

$$(1 \otimes a_{mj}) \cdot s_{00} = e^{-i\lambda mj} s_{-m,j} = (e^{-i\lambda mj} a_{-m,j} \otimes 1) \cdot s_{00},$$

from which we conclude that \mathbf{S} is (isomorphic to) the modulation of the homomorphism

$$(10) \quad S : A^{\text{op}} \rightarrow A, \quad S(a_{mj}) = e^{-i\lambda mj} a_{-m,j}.$$

Note, that $S^2 = \text{id}$ as would be the case for a cocommutative Hopf algebra, and that $S \circ * \circ S \circ * = \text{id}$ as expected. However, it turns out that axiom (H3) does not hold for \mathbf{S} . In fact, we have the following proposition:

Proposition 2. *The sesquiunital sesquialgebra (A, ϵ, Δ) does not admit a hopfish antipode.*

Proof. Any isomorphism $\psi : \mathbf{S}^* \rightarrow \text{Hom}_A(\epsilon, \Delta)$ of right $A \otimes A$ -modules is a fortiori an isomorphism of right $A \otimes 1$ -modules. Such an isomorphism must map eigenvectors of algebra elements to eigenvectors. Since any hopfish antipode \mathbf{S} is isomorphic to A as a left $A \otimes \mathbb{C} = A$ -module, its dual \mathbf{S}^* is isomorphic to A^* as a right A -module. Consider $z \in A^*$ defined by $z(a_{nl}) = \delta_{n,0}$. We have

$$(11) \quad (z \cdot a_{01})(a_{nl}) = z(a_{01} * a_{nl}) = z(a_{n,l+1}) = \delta_{n,0} = z(a_{nl}),$$

that is, z is an eigenvector of a_{01} . We will now show that this eigenvector cannot be mapped to an eigenvector and conclude that ψ cannot be an isomorphism.

Let us first determine the $A \otimes A$ -module structure of $\text{Hom}_A(\epsilon, \Delta)$ explicitly. Relabel the basis (5) of Δ by

$$\tilde{d}_{nn_2l} := d_{n_2+n, n_2, l}, \quad n, n_2, l \in \mathbb{Z},$$

that is, we substitute $n := n_1 - n_2$. The right action (6) of A now reads

$$\tilde{d}_{nn_2l} \cdot a_{mj} = e^{i\lambda m(l-j)} \tilde{d}_{n, n_2+m, l-j}.$$

From this we can see that, as a right A -module, Δ is the direct sum of the modules

$$\Delta \cong \bigoplus_{n \in \mathbb{Z}} V_n, \quad V_n := \text{Span}_{\mathbb{C}}\{\tilde{d}_{nn_2l} \mid n_2, l \in \mathbb{Z}\}.$$

Each A -module V_n is simple, cyclic, generated by $\tilde{d}_{n00} = d_{n00}$, and free. Hence, Δ is a free right A -module with A -basis $D := \{d_{n00} \mid n \in \mathbb{Z}\}$. We deduce that we have isomorphisms of vector spaces³

$$\text{Hom}_A(\epsilon, \Delta) \cong \text{Hom}_{\text{Set}}(\epsilon, D).$$

That is, every homomorphism in $\zeta \in \text{Hom}_A(\epsilon, \Delta)$ is determined by its values on d_{n00} , which can be chosen freely. Such a homomorphism can, therefore, be represented by the matrix elements $\zeta_n^l \in \mathbb{C}$ defined as

$$(12) \quad \zeta(d_{n00}) = \sum_l \zeta_n^l e_l,$$

where $\{e_l\}$ is the basis of ϵ defined in Eq. (8). The sum over l must be finite for each n .

The right action of $a \otimes b \in A \otimes A$ on $\zeta \in \text{Hom}_A(\epsilon, \Delta)$ is by pullback:

$$(\zeta \cdot (a \otimes b))(d) := \zeta((a \otimes b) \cdot d), \quad d \in \Delta.$$

³Recall our convention that $\text{Hom}(X, Y)$ denotes the set of homomorphisms to X from Y .

For the action of the basis of $A \otimes \mathbb{C}$, we evaluate

$$(\zeta \cdot (a_{mj} \otimes 1))(d_{n00}) = e^{-i\lambda j(n+m)} \zeta(d_{n+m,0,0}) \cdot a_{0j},$$

where the action of the right hand side is the right action of A on ϵ . This can be expressed in terms of the matrix elements as

$$(13) \quad (\zeta \cdot (a_{mj} \otimes 1))_n^l = e^{-i\lambda j(n+m)} \zeta_{n+m}^{l+j}.$$

Now we come back to the eigenvector z defined in Eq. (11). Let $\zeta := \psi(z) \in \text{Hom}_A(\epsilon, \Delta)$. If ψ is right $A \otimes 1$ -linear we have $\zeta \cdot (a_{01} \otimes 1) = \zeta$. Assume that there is a non-zero matrix element ζ_n^l . By Eq. (13), we have $\zeta_n^l = (\zeta \cdot (e_{01} \otimes 1))_n^l = \exp(-\lambda i n) \zeta_n^{l+1}$. It follows by induction that ζ_n^k is nonzero for all $k \geq l$. But this contradicts the fact that for fixed n the sum $\sum \zeta_n^l e_l \in \epsilon$ is finite. Hence, ζ must be zero, so the kernel of ψ is not empty. \square

3.3. The weak hopfish antipode. The nonexistence of a hopfish antipode is tied to the algebraic dual appearing in the axiom (H3) for the preantipode. For infinite dimensional vector spaces, forming the algebraic dual is an inconvenient operation, as it raises the dimension (i.e. the cardinality of a basis). Thus, infinite dimensional vector spaces are never reflexive, and countably infinite spaces do not have a predual at all. This suggests replacing axiom (H3) by a weaker notion of preantipode which does not involve the dual.

Such a weaker definition of preantipode is obtained by substituting the notion of duality in the definition of the preantipode with that of a dual pairing. Recall that a dual pairing of complex vector spaces U and V is a \mathbb{C} -bilinear map $\langle \cdot, \cdot \rangle : U \times V \rightarrow \mathbb{C}$. Let A be a unital ring. If U is a right A -module and V is a left A -module, the pairing is called A -tensorial if $\langle u \cdot a, v \rangle = \langle u, a \cdot v \rangle$ for all $u \in U$, $v \in V$, and $a \in A$. If, for each $u \in U$, the vanishing of $\langle u, v \rangle = 0$ for all $v \in V$ implies $u = 0$, then the pairing is called non-degenerate in U . Nondegeneracy in V is defined similarly, and the pairing is simply called non-degenerate if it is non-degenerate in both U and V .

Definition 3. Let \mathbf{S} be a left $A \otimes A$ -module. \mathbf{S} is called a weak hopfish preantipode, if the following holds:

(H3') There is a non-degenerate $A \otimes A$ -tensorial dual pairing of $\text{Hom}_A(\epsilon, \Delta)$ and \mathbf{S} .

A weak hopfish preantipode is called a weak hopfish antipode if it satisfies axiom (H4).

This is really a weaker notion of antipode, since, with respect to the canonical pairing of a module with its algebraic dual, every hopfish preantipode is a weak preantipode and every hopfish antipode a weak hopfish antipode.

In the case of the irrational rotation algebra, we have already seen that \mathbf{S} satisfies axiom (H4). It is isomorphic to the modulation of the homomorphism S defined in Eq. (10), the isomorphism being induced by identifying s_{00} with $a_{00} = 1_A$. Hence, for every element $s \in \mathbf{S}$ there is a unique $a_s \in A$ such that $(a_s \otimes 1) \cdot s_{00} = s$.

Theorem 1. \mathbf{S} is a weak hopfish antipode with respect to the pairing defined for $\zeta \in \text{Hom}_A(\epsilon, \Delta)$ and $s \in \mathbf{S}$ by

$$(14) \quad \langle \zeta, s \rangle := (\zeta \cdot (a_s \otimes 1))_0^0,$$

where the right hand side denotes the matrix element defined in Eq. (12).

Proof. First of all, we compute, as in the derivation of Eq. (13), the action of $A \otimes A$ on the matrix elements of $\zeta \in \text{Hom}_A(\epsilon, \Delta)$:

$$(15) \quad (\zeta \cdot (e_{m_1 j_1} \otimes e_{m_2 j_2}))_n^l = e^{-i\lambda j_1(n+m_1-m_2)} e^{im_2 l} \zeta_{n+m_1-m_2}^{l+j_1+j_2}.$$

We need to show $A \otimes A$ -tensoriality of the pairing. Note, that the map $s \mapsto a_s$ is by construction left A -linear, $a_{(b \otimes 1) \cdot s} = b * a_s$. Hence,

$$\langle \zeta, (b \otimes 1) \cdot s \rangle = (\zeta \cdot ([b * a_s] \otimes 1))_0^0 = ([\zeta \cdot (b \otimes 1)] \cdot (a_s \otimes 1))_0^0 = \langle \zeta \cdot (b \otimes 1), s \rangle,$$

so the pairing is $A \otimes \mathbb{C}$ -tensorial. Using Eq. (15) we obtain

$$(\zeta \cdot (S(a_{m,j}) \otimes 1))_0^0 = (\zeta \cdot (e^{-i\lambda m j} a_{-m,j} \otimes 1))_0^0 = \zeta_{-m}^j = (\zeta \cdot (1 \otimes a_{mj}))_0^0,$$

which implies

$$\langle \zeta \cdot (S(b) \otimes 1), s_{00} \rangle = \langle \zeta \cdot (1 \otimes b), s_{00} \rangle.$$

Furthermore, since \mathbf{S} is the modulation of S , we have $(1 \otimes b) \cdot s = ([a_s * S(b)] \otimes 1) \cdot s_{00}$. We conclude

$$\begin{aligned} \langle \zeta, (1 \otimes b) \cdot s \rangle &= \langle \zeta, ([a_s * S(b)] \otimes 1) \cdot s_{00} \rangle = \langle \zeta \cdot ([a_s * S(b)] \otimes 1), s_{00} \rangle \\ &= \langle \zeta \cdot (a_s \otimes b), s_{00} \rangle = \langle \zeta \cdot (1 \otimes b), s \rangle, \end{aligned}$$

that is, the pairing is $\mathbb{C} \otimes A$ tensorial.

It remains to show that the pairing is non-degenerate. Given $\zeta \in \text{Hom}_A(\epsilon, \Delta)$, assume that $\langle \zeta, s \rangle = 0$ for all $s \in \mathbf{S}$. Then $(\zeta \cdot (a \otimes 1))_0^0 = 0$ for all $a \in A$. By Eq. (15) we get

$$(\zeta \cdot (a_{nl} \otimes 1))_0^0 = e^{-i\lambda nl} \zeta_n^l = 0,$$

for all $n, l \in \mathbb{Z}$. Hence, $\zeta = 0$. Given $s \in \mathbf{S}$ assume now that $\langle \zeta, s \rangle = 0$ for all $\zeta \in \text{Hom}_A(\epsilon, \Delta)$. Write $a_s = \sum_{nl} \alpha^{nl} e_{nl}$ with coefficients $\alpha^{nl} \in \mathbb{C}$. Let now $\zeta \in \text{Hom}_A(\epsilon, \Delta)$ be the map with only one non-zero matrix element $\zeta_n^l = 1$. Then

$$\langle \zeta, s \rangle = e^{-i\lambda nl} \alpha^{nl} = 0.$$

By choosing such a ζ for all $n, l \in \mathbb{Z}$ we conclude that all α^{nl} s vanish, so $a = 0$. \square

4. THE TENSOR PRODUCT OF MODULES

The coproduct bimodule Δ determines a tensor product operation \otimes_Δ on right A -modules T and T' defined by

$$(16) \quad T \otimes_\Delta T' := (T \otimes T') \otimes_{A \otimes A} \Delta.$$

By the axioms for a sesquiunital sesquialgebra, the tensor product descends to a monoid structure on the isomorphism classes of right A -modules. For the tensor product on isomorphism classes, Axiom (H1) implies associativity up to isomorphism and Axiom (H2) implies that the counit ϵ is the unit element.

For simplicity of notation, in this section we shall write the convolution product of $a, b \in A$ as ab , without the star.

4.1. Simple modules generated by an eigenvector of a unitary. We now consider a class of simple right $*$ -modules of A generated by an eigenvector of a unitary element $u \in U(A)$. Since the eigenvalue z must be in $U(1)$ we can rescale the eigenvalue to 1 by choosing the unitary $z^{-1}u$ instead. The most obvious examples of such modules are obtained as quotient of A by the right ideal $(u - 1)A$, which is the smallest possible annihilator of an eigenvector of u with eigenvalue 1,

$$(17) \quad T = A/(u - 1)A.$$

We will prove that such a module is simple if and only if u does not have roots.

Lemma 1. *The invertible (unitary) elements of A are the invertible (unitary) scalar multiples of the basis elements,*

$$A^\times = \{\mu a_{pq} \mid \mu \in \mathbb{C}^\times, p, q \in \mathbb{Z}\}, \quad U(A) = \{\mu a_{pq} \mid \mu \in \mathbb{C}^\times, |\mu| = 1, p, q \in \mathbb{Z}\}.$$

Furthermore, A is a division ring.

Proof. A is graded with respect to the \mathbb{Z}^2 -grading given by $\deg(a_{pq}) = (p, q)$. Equip \mathbb{Z}^2 with the lexicographical ordering. For a general nonzero $a \in A$, the maximal and minimal degrees of its nonzero homogeneous terms are denoted by $\deg_{\max}(a)$ and $\deg_{\min}(a)$ respectively; the degree of 0 is denoted by \emptyset and considered to be less than any other degree. Now $\deg_{\max}(ab) = \deg_{\max}(a) + \deg_{\max}(b)$ and $\deg_{\min}(ab) = \deg_{\min}(a) + \deg_{\min}(b)$. Now let a be invertible with inverse b . Since $\deg(1) = (0, 0)$, $ab = 1$ implies $\deg_{\max}(a) = -\deg_{\max}(b)$ and $\deg_{\min}(a) = -\deg_{\min}(b)$. Since furthermore $\deg_{\max}(a) \geq \deg_{\min}(a)$ it follows that $\deg_{\max}(a) = \deg_{\min}(a)$. Hence, a is of homogeneous degree, that is, proportional to a basis element, and all basis elements are unitary. By a similar reasoning it follows from $ab = 0$ and $\deg(0) = \emptyset$ that a and b cannot both be nonzero. \square

Proposition 3. *If $u \in U(A)$ does not have any roots then the right A -module $A/(u - 1)A$ is simple.*

Proof. To prove that $A/(u - 1)A$ is simple, we prove that if $v \notin (u - 1)A$, then $I := vA + (u - 1)A$ is equal to A . We start with several observations.

- (1) u can not be a constant because otherwise u has an n 'th root for every n . Therefore $u = \mu a_{pq}$ with $\mu \in U(1)$, $(p, q) \neq (0, 0)$.
- (2) If $\gcd(p, q) = d > 1$, then $\mu a_{pq} = (\sqrt[d]{\mu} e^{-\frac{i\lambda pq(d-1)}{2d^2}} a_{\frac{p}{d}, \frac{q}{d}})^d$. Therefore, p and q have to be relatively prime.
- (3) Since $\gcd(p, q) = 1$, there is $(r, s) \in \mathbb{Z} \times \mathbb{Z}$ such that (p, q) and (r, s) form a basis of $\mathbb{Z} \times \mathbb{Z}$. Therefore, a_{pq} and a_{rs} generate A , and $v = \sum_{k,l} \alpha_{k,l} a_{pq}^k a_{rs}^l$. Because $a_{pq}^k - 1 = (a_{pq} - 1)(a_{pq}^{k-1} + \dots + 1) \in (u - 1)A$, we can assume that $v = \sum_l \alpha_l a_{rs}^l \neq 0$, a Laurent polynomial in a_{rs} . Furthermore, some α_l is non-zero and a_{rs}^l is invertible, so $v(a_{rs}^l)^{-1}$ generates the same ideal as v and has a non-zero constant term. Hence, we can assume without loss of generality that v is a polynomial with a non-zero constant term.

Now we show that I is equal to A by proving that $1 \in I$. Assume that v be a polynomial of degree n . Since $a_{pq} a_{rs}^n a_{pq}^{-1} = e^{-i\lambda nps} a_{rs}^n$, when $n > 0$, $v' = e^{-i\lambda nps} u v u^{-1} - v = (u - 1) e^{-i\lambda nps} v u^{-1} + v(e^{-i\lambda nps} u^{-1} - 1) \in I$ is a polynomial of degree $n - 1$ with a constant term equal to 1. By repeating this construction, we conclude that 1 is in I . \square

Proposition 4. *If $u \in U(A)$ has a primitive d -th root $\sqrt[d]{u}$ then $A/(u-1)A$ can be decomposed as*

$$(18) \quad A/(u-1)A \cong \bigoplus_{n=0}^{d-1} A/(e^{-\frac{2\pi i n}{d}} \sqrt[d]{u} - 1)A$$

into a direct sum of simple right A -modules.

Proof. All primitive d -th roots of u can be obtained by multiplying $\sqrt[d]{u}$ with a d -th root of unity in \mathbb{C} , so for every integer n , $0 \leq n \leq d-1$ we can factorize

$$(19) \quad u - 1 = \left(\sum_{k=0}^{d-1} (e^{-\frac{2\pi i n}{d}} \sqrt[d]{u})^k \right) (e^{-\frac{2\pi i n}{d}} \sqrt[d]{u} - 1).$$

Let $\xi := 1 + (u-1)A$ be the canonical cyclic vector of $A/(u-1)A$. Define for all integers $0 \leq n \leq d-1$ the vectors

$$\xi_n := \xi \cdot \frac{1}{\sqrt[d]{d}} \sum_{k=0}^{d-1} (e^{-\frac{2\pi i n}{d}} \sqrt[d]{u})^k.$$

Consider the cyclic submodule $\xi_n \cdot A$ which is isomorphic to $A/\text{Ann}(\xi_n)$. By Eq. (19) the annihilator of ξ_n contains the ideal $I := (e^{-\frac{2\pi i n}{d}} \sqrt[d]{u} - 1)A$. Since $\sqrt[d]{u}$ is a primitive root, it does not have proper roots itself, so it follows from Proposition 3 that I is maximal and, hence, equal to the annihilator of ξ_n . We conclude that

$$\xi_n \cdot A \cong A/(e^{-\frac{2\pi i n}{d}} \sqrt[d]{u} - 1)A.$$

The cyclic vector can be retrieved as $\xi := \frac{1}{\sqrt[d]{d}} \sum_{n=0}^{d-1} \xi_n$, which implies that the ξ_n 's generate the whole module,

$$A/(u-1)A = \xi_0 \cdot A + \dots + \xi_{d-1} \cdot A.$$

It remains to show that this sum is direct. Since $\{a_{jk} \mid j, k \in \mathbb{Z}\}$ is a basis of A , the set of vectors $\{\xi_n \cdot a_{jk} \mid j, k \in \mathbb{Z}\}$ spans $\xi_n \cdot A$. The primitive root is of the form $\sqrt[d]{u} = e^{-i\alpha} a_{pq}$. Using Eq. (2) for the multiplication in A we get $(\xi_n \cdot a_{jk}) \cdot \sqrt[d]{u} = e^{i[\frac{2\pi n}{d} + \lambda(jq - kp)]} (\xi_n \cdot a_{jk})$. Hence, each summand $\xi_n \cdot A$ can be decomposed into eigenspaces of $\sqrt[d]{u}$ with eigenvalues of the form $e^{i[\frac{2\pi n}{d} + \lambda k]}$ for some integer k . Because $\frac{\lambda}{2\pi}$ is irrational, the eigenvalues of the eigenspace decomposition of one summand $\xi_n \cdot A$ are all different from those of every other summand $\xi_{n'} \cdot A$, $n' \neq n$. We conclude that the subspaces $\xi_0 \cdot A, \dots, \xi_{d-1} \cdot A$ are linearly independent, so their sum is direct. \square

Corollary 1. *The right A -module $A/(u-1)A$ is simple if and only if $u = e^{-i\alpha} a_{pq}$ for $\alpha \in \mathbb{R}$ and $p, q \in \mathbb{Z}$ relatively prime.*

Proof. From Eq. (2) we derive for all $d, p', q' \in \mathbb{Z}$ and $p = dp', q = dq'$:

$$(20) \quad (a_{p'q'})^d = e^{\frac{i\lambda p'q'd(d-1)}{2}} a_{dp', dq'} \Leftrightarrow \sqrt[d]{a_{pq}} = e^{-\frac{i\lambda pq(d-1)}{2d^2}} a_{\frac{p}{d}, \frac{q}{d}}.$$

This shows that a_{pq} does not have roots if and only if p and q are relatively prime. \square

Definition 4. *Let $\alpha \in \mathbb{R}$, $p, q \in \mathbb{Z}$ relatively prime. Define*

$$T_{pq}^\alpha := A/(e^{-i\alpha} a_{pq} - 1)A.$$

4.2. Construction of a basis. We will find a basis for the simple modules T_{pq}^α . Let $u := e^{-i\alpha}a_{pq}$ and let $\xi := 1 + (u - 1)A$ be the canonical cyclic vector. The vectors $\xi'_{jk} := \xi \cdot a_{jk}$ span T_{pq}^α . Using Eq. (2), we get on the one hand,

$$(21) \quad \xi'_{jk} \cdot u = e^{i\lambda(jq-kp)} \xi'_{jk},$$

on the other hand, $\xi'_{jk} \cdot u = \xi \cdot (e^{i(-\alpha+\lambda jq)}a_{j+p,k+q}) = e^{i(-\alpha+\lambda jq)}\xi_{j+p,k+q}$, which implies

$$\xi_{j+p,k+q} = e^{i(\alpha-\lambda kp)} \xi'_{jk}.$$

We need to consider two cases separately.

Case 1: $p \neq 0$. We can rescale the vectors ξ'_{jk} to

$$\xi_{jk} := e^{\frac{i}{p}\{-\alpha j + \lambda[jkp - \frac{1}{2}j(j+p)q]\}} \xi'_{jk},$$

which satisfy $\xi_{jk} = \xi_{j+p,k+q}$. Hence, the vector ξ_{jk} can be labeled by $[j, k] := (j, k) + (p, q)\mathbb{Z}$. Since p and q are relatively prime, there is a bijection

$$(22) \quad \nu : \mathbb{Z}^2 / (p, q)\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}, \quad [j, k] \mapsto jq - kp.$$

Eq. (21) implies that, if $[j, k] \neq [j', k']$ then $\xi_{[j,k]}$ and $\xi_{[j',k']}$ are eigenvectors of u with different eigenvalues. We conclude that the set of vectors

$$(23) \quad B := \{\xi_{[j,k]} \mid [j, k] \in \mathbb{Z}^2 / (p, q)\mathbb{Z}\}$$

is an orthonormal basis of T_{pq}^α . The action of the generators of A on this basis is

$$(24) \quad \begin{aligned} \xi_{[j,k]} \cdot a_{10} &= e^{\frac{i}{p}[\alpha + \lambda(jq-kp) + \frac{1}{2}\lambda q(p+1)]} \xi_{[j+1,k]}, \\ \xi_{[j,k]} \cdot a_{01} &= \xi_{[j,k+1]}. \end{aligned}$$

If we label the basis instead by \mathbb{Z} via the bijection defined in Eq. (22), $\xi_n := \xi_{\nu^{-1}(n)}$, the action reads

$$(25) \quad \begin{aligned} \xi_n \cdot a_{10} &= e^{\frac{i}{p}[\alpha + \lambda n + \frac{1}{2}\lambda q(p+1)]} \xi_{n+q}, \\ \xi_n \cdot a_{01} &= \xi_{n-p}. \end{aligned}$$

Case 2: $p = 0$. Because the module T_{0q}^α is assumed to be simple, q must be nonzero. We can rescale

$$\xi_{jk} := e^{-\frac{i\alpha k}{q}} \xi'_{jk},$$

such that $\xi_{jk} = \xi_{j+p,k+q}$. The set of vectors defined in Eq. (23) is again an orthonormal basis. The action of the generators of A on this basis is

$$(26) \quad \begin{aligned} \xi_{[j,k]} \cdot a_{10} &= \xi_{[j+1,k]}, \\ \xi_{[j,k]} \cdot a_{01} &= e^{\frac{i}{q}[\alpha + \lambda jq]} \xi_{[j,k+1]}. \end{aligned}$$

Again, labeling the basis by \mathbb{Z} , we read the action to be

$$(27) \quad \begin{aligned} \xi_n \cdot a_{10} &= \xi_{n+q}, \\ \xi_n \cdot a_{01} &= e^{\frac{i}{q}[\alpha + \lambda n]} \xi_n. \end{aligned}$$

We can use these formulas to construct modules in the case that p and q are not relatively prime:

Definition 5. Let T be the inner product space spanned by the orthonormal basis $\{\xi_n \mid n \in \mathbb{Z}\}$. Let $\alpha \in \mathbb{R}$, $p, q \in \mathbb{Z}$ where $p \neq 0$ or $q \neq 0$. The right action of A on T given by Eqs. (25) for $p \neq 0$ and by Eqs. (27) for $p = 0$ defines a right A $*$ -module, which we denote by T_{pq}^α .

4.3. Isomorphism classes of the modules. We conclude the general study of the modules T_{pq}^α by giving a criterion for two modules to be isomorphic.

Proposition 5. *The modules T_{pq}^α and T_{rs}^β are isomorphic iff $(p, q) = \pm(r, s)$ and $\alpha = \pm\beta + n\lambda$ for some $n \in \mathbb{Z}$.*

Proof. First, let us assume that T_{pq}^α and T_{rs}^β are simple. By Eq. (25), the matrix with respect to the basis $\{\xi_n\}$ for a_{rs} acting on T_{pq}^α has all its nonzero elements on a diagonal which is shifted from the main diagonal by $rq - sp$ units. Unless $rq - sp = 0$, this matrix has no eigenvectors. Since all the ξ_n are eigenvalues for the action of a_{pq} on T_{pq}^α , it follows that the two modules in question can be isomorphic if and only if the integer vectors (p, q) and (r, s) are collinear. Now $\gcd(p, q) = \gcd(r, s) = 1$ implies that $(p, q) = \pm(r, s)$. The statement about α and β follows from a comparison of the eigenvalues. This completes the proof of the “only if” part of our proposition. The “if” part follows immediately from the definition of the modules.

Now if a_{pq} has a primitive d -th root, T_{pq}^α is the direct sum of the modules generated by each of $\xi_k, \xi_{k+1}, \dots, \xi_{k+d-1}$ for some $k \in \mathbb{Z}$. For convenience we choose $k := -\frac{pq(d-1)}{2d}$. Labelling the basis of $\xi_{k+l} \cdot A$ as $\eta_j := \xi_{k+l+dj}$, we can read off Eqs. (25) and (27) that $\xi_{k+l} \cdot A \cong T_{p/d, q/d}^{[(\alpha+\lambda l)/d]}$. We thus obtain the decomposition into simple modules

$$T_{pq}^\alpha \cong \bigoplus_{l=0}^{d-1} T_{\frac{p}{d}, \frac{q}{d}}^{[\frac{\alpha+\lambda l}{d}]}.$$

For T_{pq}^α and T_{rs}^β to be isomorphic the simple modules of this decomposition have to be pairwise isomorphic. This is the case if there are $0 \leq l, m < d$ and $n \in \mathbb{Z}$ such that $(\alpha + \lambda l)/d = \pm(\beta + \lambda m)/d + \lambda n \Leftrightarrow \alpha = \pm\beta + \lambda n'$, where $n' = dn - l \pm m$. \square

4.4. Calculation of the tensor product. We proceed to calculate the tensor product of two modules $T_{p_1 q_1}^{\alpha_1}$ and $T_{p_2 q_2}^{\alpha_2}$ of Definition 5. As a vector space we have

$$T_{p_1 q_1}^{\alpha_1} \otimes_{\Delta} T_{p_2 q_2}^{\alpha_2} = \text{Span}\{\xi_{k_1}^1 \otimes \xi_{k_2}^2 \otimes d_{n_1 n_2 l} \mid k_1, k_2, n_1, n_2, l \in \mathbb{Z}\} / V,$$

where $\xi_{k_1}^1$ and $\xi_{k_2}^2$ are the basis vectors of $T_{p_1 q_1}^{\alpha_1}$ and $T_{p_2 q_2}^{\alpha_2}$ from Definition 5, and where V is the ideal (the vector space) generated by the relations

$$(\xi_{k_1}^1 \cdot a_1) \otimes (\xi_{k_2}^2 \cdot a_2) \otimes d_{n_1 n_2 l} = \xi_{k_1}^1 \otimes \xi_{k_2}^2 \otimes (a_1 \otimes a_2) \cdot d_{n_1 n_2 l},$$

for all $a_1, a_2 \in A$. Using $d_{n_1 n_2 l} = e^{-i\lambda n_1 l}(a_{n_1, -l} \otimes a_{n_2 0}) \cdot d_{000}$ we obtain that $T_{p_1 q_1}^{\alpha_1} \otimes_{\Delta} T_{p_2 q_2}^{\alpha_2}$ is spanned by vectors of the form

$$\xi'_{k_1, k_2} := \xi_{k_1}^1 \otimes \xi_{k_2}^2 \otimes d_{000}.$$

These vectors are not yet linearly independent. We still have to consider the relation generated by the action of $1 \otimes a_{01}$. Using $(1 \otimes a_{01}) \cdot d_{000} = d_{00, -1} = (a_{01} \otimes 1) \cdot d_{000}$, we obtain the relation

$$(28) \quad \xi_{k_1}^1 \otimes (\xi_{k_2}^2 \cdot a_{01}) \otimes d_{000} = (\xi_{k_1}^1 \cdot a_{01}) \otimes \xi_{k_2}^2 \otimes d_{000}.$$

We have to distinguish three cases:

Case 1: $p_1 \neq 0$ and $p_2 \neq 0$. Under this assumption relation (28) becomes

$$\xi'_{k_1+p_1, k_2-p_2} = \xi'_{k_1, k_2}.$$

We conclude that we can label these vectors uniquely by $\mathbb{Z}^2/(p_1, -p_2)\mathbb{Z}$,

$$\xi_{[k_1, k_2]} := \xi'_{k_1, k_2}, \quad [k_1, k_2] := (k_1, k_2) + (p_1, -p_2)\mathbb{Z}.$$

By construction, these vectors form a basis of the tensor product module. The action of $a_{10} \in A$ on this basis is given by

$$\begin{aligned} \xi_{[k_1, k_2]} \cdot a_{10} &= \xi_{k_1}^1 \otimes \xi_{k_2}^2 \otimes [d_{000} \cdot a_{10}] \\ &= \xi_{k_1}^1 \otimes \xi_{k_2}^2 \otimes [(a_{10} \otimes a_{10}) \cdot d_{000}] \\ &= (\xi_{k_1}^1 \cdot a_{10}) \otimes (\xi_{k_2}^2 \cdot a_{10}) \otimes d_{000}, \end{aligned}$$

and similarly for a_{01} . Inserting (25) yields

$$\begin{aligned} (29) \quad \xi_{[k_1, k_2]} \cdot a_{10} &= e^{i\left\{\frac{\alpha_1 + \lambda[k_1 + q_1(p_1 + 1)/2]}{p_1} + \frac{\alpha_2 + \lambda[k_2 + q_2(p_2 + 1)/2]}{p_2}\right\}} \xi_{[k_1 + q_1, k_2 + q_2]} \\ \xi_{[k_1, k_2]} \cdot a_{01} &= \xi_{[k_1 - p_1, k_2]}. \end{aligned}$$

Using the bijection

$$\begin{aligned} \nu : \mathbb{Z}^2/(p_1, -p_2)\mathbb{Z} &\xrightarrow{\cong} (\mathbb{Z}/\gcd(p_1, p_2)\mathbb{Z}) \times \mathbb{Z} \\ [k_1, k_2] &\mapsto \left(q_2 k_1 - q_1 k_2 \bmod \gcd(p_1, p_2)\mathbb{Z}, \frac{p_2 k_1 + p_1 k_2}{\gcd(p_1, p_2)} \right), \end{aligned}$$

we relabel the basis by setting $\xi_n^{(m)} := \xi_{\nu^{-1}(m, n)}$. The action (29) then takes the form

$$\begin{aligned} (30) \quad \xi_n^m \cdot a_{10} &= e^{\frac{i}{p}[\alpha + \lambda n + \frac{1}{2}q(p+1)]} \xi_{n+q}^m \\ \xi_n^m \cdot a_{01} &= \xi_{n-p}^m, \end{aligned}$$

where

$$p := \text{lcm}(p_1, p_2), \quad q := \frac{p_1 q_2 + p_2 q_1}{\gcd(p_1, p_2)}, \quad \alpha := \frac{\alpha_1 p_2 + \alpha_2 p_1}{\gcd(p_1, p_2)} + \frac{\lambda p(q_1 + q_2 - q)}{2}.$$

Comparing this action with (25) we infer

$$(31) \quad T_{p_1 q_1}^{\alpha_1} \otimes_{\Delta} T_{p_2 q_2}^{\alpha_2} \cong \gcd(p_1, p_2) T_{pq}^{\alpha},$$

where the prefactor on the right hand side denotes the direct sum of $\gcd(p_1, p_2)$ copies of the same module.

Case 2: either $p_1 = 0$ or $p_2 = 0$. The calculation of the tensor product for this case is very similar to the preceding case. The result is again given by Eq. (31).

Case 3: $p_1 = 0$ and $p_2 = 0$. This case is quite different. Using Eq. (27) for the action of a_{01} , relations (28) become

$$\xi_{k_1}^1 \otimes (e^{\frac{i}{q_2}[\alpha_2 + \lambda k_2]} \xi_{k_2}^2) \otimes d_{000} = (e^{\frac{i}{q_1}[\alpha_1 + \lambda k_1]} \xi_{k_1}^1) \otimes \xi_{k_2}^2 \otimes d_{000},$$

which is equivalent to

$$\left(e^{\frac{i}{q_2}[\alpha_2 + \lambda k_2]} - e^{\frac{i}{q_1}[\alpha_1 + \lambda k_1]} \right) \xi'_{k_1, k_2} = 0.$$

It follows that ξ'_{k_1, k_2} is zero unless

$$\frac{\alpha_1 q_2 - \alpha_2 q_1 + \lambda(k_1 q_2 - k_2 q_1)}{q_1 q_2} \cong 0 \bmod 2\pi\mathbb{Z}.$$

Assuming (without loss of generality) that $0 \leq \alpha_1, \alpha_2 < 2\pi$ and using that $\frac{\lambda}{2\pi}$ is irrational, we conclude that all ξ'_{k_1, k_2} vanish unless

$$r := -\frac{\alpha_1 q_2 - \alpha_2 q_1}{\lambda \gcd(q_1, q_2)}$$

is an integer modulo integer multiples of $\text{lcm}(q_1, q_2) \frac{2\pi}{\lambda}$.

In this case a basis for the tensor module is given by

$$B := \left\{ \xi'_{k_1, k_2} \mid k_1, k_2 \in \mathbb{Z}, \frac{k_1 q_2 - k_2 q_1}{\gcd(q_1, q_2)} = r \right\}.$$

We can relabel the basis by \mathbb{Z} : Let s_1, s_2 be integers such that

$$(32) \quad \frac{s_1 q_2 - s_2 q_1}{\gcd(q_1, q_2)} = 1.$$

This can be used to construct a bijection

$$\begin{aligned} \nu : \mathbb{Z} &\xrightarrow{\cong} \left\{ (k_1, k_2) \in \mathbb{Z} \mid \frac{k_1 q_2 - k_2 q_1}{\gcd(q_1, q_2)} = r \right\}, \\ n &\longmapsto r(s_1, s_2) + \frac{k}{\gcd(q_1, q_2)}(q_1, q_2), \end{aligned}$$

and relabel the basis by $\xi_n := \xi'_{\nu(n)}$. In terms of the relabeled basis the action of the generators a_{10}, a_{01} takes the form of Eqs. (27) with

$$q := \gcd(q_1, q_2), \quad \alpha := s_1 \alpha_2 - s_2 \alpha_1.$$

For convenience we summarize the results obtained in this section.

Theorem 2. *Let $T_{p_1 q_1}^{\alpha_1}$ and $T_{p_2 q_2}^{\alpha_2}$ be the right A -modules of Definition 5, their tensor product being defined in Eq. (16).*

For $p_1 \neq 0$ or $p_2 \neq 0$ we have:

$$T_{p_1 q_1}^{\alpha_1} \otimes_{\Delta} T_{p_2 q_2}^{\alpha_2} \cong \gcd(p_1, p_2) T_{pq}^{\alpha},$$

where

$$(33) \quad p := \text{lcm}(p_1, p_2), \quad q := \frac{p_1 q_2 + p_2 q_1}{\gcd(p_1, p_2)}, \quad \alpha := \frac{\alpha_1 p_2 + \alpha_2 p_1}{\gcd(p_1, p_2)}.$$

For $p_1 = 0$ and $p_2 = 0$ we have:

$$T_{0, q_1}^{\alpha_1} \otimes_{\Delta} T_{0, q_2}^{\alpha_2} \cong \begin{cases} T_{0, q}^{\alpha}, & \text{for } \frac{\alpha_1 q_2 - \alpha_2 q_1}{\lambda \gcd(q_1, q_2)} \in \mathbb{Z} \bmod \text{lcm}(q_1, q_2) \frac{2\pi}{\lambda}, \\ 0, & \text{otherwise} \end{cases},$$

where

$$(34) \quad q := \gcd(q_1, q_2), \quad \alpha := s_1 \alpha_2 - s_2 \alpha_1, \quad s_1, s_2 \in \mathbb{Z} : \quad \frac{s_1 q_2 - s_2 q_1}{\gcd(q_1, q_2)} = 1.$$

We end this paper with the following remarks.

Remark 1. We have dropped the term $\frac{\lambda p(q_1 + q_2 - q)}{2}$ in the expression (33) for α because $p(q_1 + q_2 - q)$ is always even.

Remark 2. The formulas for \otimes_{Δ} in Theorem 2 have the following simple interpretation as addition of fractions, e.g.

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p}{q}.$$

This is connected with the second symplectic groupoid structure mentioned in the Introduction.

Remark 3. The operation \otimes_{Δ} extends to an associative commutative product on the free abelian group \mathcal{R} generated by the T_{pq}^{α} , with the module $T_{10}^0 = \epsilon$ as unit of \mathcal{R} . The resulting unital ring \mathcal{R} seems to contain the necessary information to reconstruct the group structure on the original quotient S^1/\mathbb{Z} .

REFERENCES

- [1] Baez, John C., and Lauda, Aaron D., Higher-dimensional algebra. V. 2-groups, *Theory Appl. Categ.* **12** (2004), 423–491
- [2] Behrend, K., Cohomology of stacks. *Intersection theory and moduli*, (electronic), *ICTP Lect. Notes*, XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, 249–294
- [3] Behrend, K., and Xu, P., Differentiable stacks and gerbes, *math.DG/0605694* (2006).
- [4] Bonneau, P., Flato, M., Gerstenhaber, M., and Pinczon, G., The hidden group structure of quantum groups: strong duality, rigidity and preferred deformations, *Comm. Math. Phys.* **161** (1994), 125–156.
- [5] Khalkhali, M., and Rangipour, B., Para-Hopf algebroids and their cyclic cohomology, *Lett. Math. Phys.* **70** (2004), no. 3, 259–272.
- [6] Lu, J.-H., and Weinstein, A., Groupoïdes symplectiques doubles des groupes de Lie-Poisson, *C. R. Acad. Sci. Paris* **309** (1989), 951–954.
- [7] Mrčun, J. *Stability and invariants of Hilsum-Skandalis maps*, PhD thesis, Utrecht University, Utrecht, 1996, *math.DG/0506484*.
- [8] Rieffel, M., *Deformation quantization for actions of R^d* , *Mem. Amer. Math. Soc.* **106** (1993), no. 506.
- [9] Tang, X., Weinstein, A., and Zhu, C., Hopfish algebras, to appear in *Pacific J. Math*, *math.QA/0510421*.
- [10] Tseng, H., and Zhu, C., Integrating Lie algebroids via stacks. *Compositio Mathematica*, Volume 142 (2006), 251–270.
- [11] Weinstein, A., Symplectic groupoids, geometric quantization, and irrational rotation algebras, *Symplectic geometry, groupoids, and integrable systems, Séminaire sud-Rhodanien de géométrie à Berkeley (1989)*, P. Dazord and A. Weinstein, eds., Springer-MSRI Series (1991), 281–290.
- [12] Woronowicz, S., Twisted $SU(2)$ group, An example of a noncommutative differential calculus, *Publ. Res. Inst. Math. Sci.*, **23** (1987), 117–181.
- [13] Zakrzewski, S., Quantum and classical pseudogroups, I and II, *Comm. Math. Phys.* **134** (1990), 347–370, 371–395.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720, USA.
 JACOBS UNIVERSITY BREMEN, CAMPUS RING 1, 28759 BREMEN, GERMANY.
E-mail address: `blohmann@math.berkeley.edu`

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MO 63139, USA.
E-mail address: `xtang@math.wustl.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720, USA.
E-mail address: `alanw@math.berkeley.edu`